Determinant of Matrices, Adjoint, Cramer's Rule, Finding Inverses.

Determinant of a matrix.

X = set of square matrices. ~~det(A~~ $\det : X \to \mathbb{R}$ is a function.

2x2 $\quad A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ $\quad \det(A) = a_{11} \cdot a_{22} - a_{12} \cdot a_{21}$ $\quad \det A = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$

3x3 $\quad A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ $\quad \det(A) = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$

For a general square matrix, need method of cofactors.

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} = [a_{ij}]_{n \times n} \quad \text{a square matrix.}$$

Defn: Minor of a_{ij} is the determinant of the matrix in the form

$$\begin{bmatrix} a_{11} & \cdots & \cdots & a_{1n} \\ \vdots & a_{ij} & & \\ \vdots & & & \\ a_{n1} & \cdots & \cdots & a_{nn} \end{bmatrix}$$

(the matrix obtained by removing ith row jth column of A)

Notation: minor of $a_{ij} := M_{ij}$

Ex: $A = \begin{bmatrix} 4 & 0 & 10 \\ -1 & 2 & 3 \\ 5 & -5 & -1 \end{bmatrix}$ $\quad M_{12} = \left| \begin{bmatrix} \cancel{4} & \cancel{0} & \cancel{10} \\ -1 & \cancel{2} & 3 \\ 5 & \cancel{-5} & -1 \end{bmatrix} \right| = \det \begin{pmatrix} -1 & 3 \\ 5 & -1 \end{pmatrix} = -14.$ $\quad (*)$

Defn: $A = [a_{ij}]_{n \times n}$. Cofactor of a_{ij} is the number $(-1)^{i+j} M_{ij}$

Notation: $C_{ij} = (-1)^{i+j} M_{ij}$

Ex: For A in $(*)$, $C_{12} = (-1)^{1+2} M_{12} = (-1) \cdot (-14) = 14.$

Defn: $A = [a_{ij}]_{n \times n}$ $\quad \det(A) = a_{i1} C_{i1} + \dots + a_{in} C_{in}$ (expansion w.r.t. ith row)

$\det(A) = a_{1j} C_{1j} + \dots + a_{nj} C_{nj}$ (expansion w.r.t jth row).

Ex: $A = \begin{bmatrix} 4 & 2 & 1 \\ -2 & -6 & 3 \\ -7 & 5 & 0 \end{bmatrix}$ $\quad$ Find $\det(A)$, w.r.t i) first row
ii) second column.

Soln:

i) along first row.

$\det(A) = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$ (make calculations.)

ii) along second column

$\det(A) = a_{12}C_{12} + a_{22}C_{22} + a_{32}C_{32}$ (make calculations).

Ex: Compute the determinant of $A = \begin{bmatrix} 5 & -2 & 2 & 7 \\ 1 & 0 & 0 & 3 \\ -3 & 1 & 5 & 0 \\ 3 & -1 & -9 & 4 \end{bmatrix}$

Soln: expand it w.r.t. second row.

Defn: Let A be an $n \times n$ matrix and C_{ij} be the cofactor of a_{ij}. The matrix of cofactors from A is

$$C = \begin{bmatrix} C_{11} \cdot C_{12} & \cdots & C_{1n} \\ \vdots & & \\ C_{n1} & C_{n2} \cdots & C_{nn} \end{bmatrix}$$

The adjoint of A is C^T.

Notation: $\operatorname{adj}(A) = C^T$.

Ex: Compute the adjoint of $A = \begin{bmatrix} 4 & 2 & 1 \\ -2 & -6 & 3 \\ -7 & 5 & 0 \end{bmatrix}$

Using determinant finding inverse:

Fact: $A = [a_{ij}]_{n \times n}$ a square matrix. If A is invertible matrix then $A^{-1} = \dfrac{\operatorname{adj}(A)}{\det(A)}$

Ex: Use adjoint matrix to compute the inverse of $A = \begin{bmatrix} 4 & 2 & 1 \\ -2 & -6 & 3 \\ -7 & 5 & 0 \end{bmatrix}$

Remark: One can calculate the determinant of a square matrix.

Not a non-square matrix.

Properties of determinant

1. $A = [a_{ij}]_{n\times n}$, $c \in \mathbb{R}$. **Ex:** $A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$

   $\det(cA) = c^n \det(A)$. Consider $\det(A)$ and $\det(2A)$.

2. $A = [a_{ij}]_{n\times n}$, $B = [b_{ij}]_{n\times n}$. **Ex:** $A = \begin{bmatrix} 1 & -1 \\ 3 & 2 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}$

   $\det(AB) = \det(A) \cdot \det(B)$

Warning!! $\det(A+B) \neq \det(A) + \det(B)$ (in general).

3. $A = [a_{ij}]_{n\times n}$ if A is invertible, $\det(A^{-1}) = \dfrac{1}{\det(A)}$. Consider $A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$

4. $A = [a_{ij}]_{n\times n}$. $\det(A) = \det(A^T)$. Consider $A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$

5. $A = [a_{ij}]_{n\times n}$ if A has a zero row, then $\det(A) = 0$. $A = \begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix}$

6. $A = [a_{ij}]_{n\times n}$ if A is triangular matrix, then $\det(A) = a_{11} \cdots a_{nn}$.

   $A = \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix}$

Fact: $A = [a_{ij}]_{n\times n}$. A is invertible $\Leftrightarrow \det(A) \neq 0$.

An invertible matrix is called non-singular

(If it's not invertible, it's called singular)

Cramers Rule:

Fact: $A = [a_{ij}]_{n\times n}$. $AX = B$

$$\left[\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} \right.$$

Then $x_1 = \dfrac{\det(A_1)}{\det(A)}$, ..., $x_n = \dfrac{\det(A_n)}{\det(A)}$

$$A_i \Rightarrow \begin{bmatrix} a_{11} \cdots & \boxed{\begin{matrix} a_{1i} \\ a_{2i} \\ \vdots \\ a_{ni} \end{matrix}} & \cdots a_{1n} \\ \cdots & & \cdots \end{bmatrix}$$

↳ replaced by $\begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$

Ex: Solve the system using cramer's rule.

$$3x + y + z = 3$$
$$2x + 2y + 5z = -1$$
$$x - 3y - 4z = 2$$

H.W. ① $x + z = 1$
$y + z = 1$
$x + y = 4$

② $4x - y + 3z = 2$
$x + 5y - 2z = 3$
$3x + 2y + 4z = 6$.