Question 1. [25 pts] Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 1 & 2 & 4 \end{bmatrix}$.

(a) [10 pts] Find $det(A)$ using cofactor expansion method.

Use cofactor expansion along the 1^{st} column:

$$\det(A) = 1 \cdot (-1)^2 \cdot \begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} + 0 \cdot (-1)^3 \cdot \begin{vmatrix} 1 & 1 \\ 2 & 4 \end{vmatrix} + 1 \cdot (-1)^4 \cdot \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix}$$

$$= 0 + 0 + 1 \cdot 1$$

$$\boxed{\det A = 1}$$

(b) [15 pts] Find A^{-1}, if A is invertible.

$\det A = 1 \Rightarrow$ A^{-1} exists (A is invertible)

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 1 & 2 & 4 & 0 & 0 & 1 \end{array}\right] \xrightarrow{-r_1 + r_3 \to r_3} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 1 & 3 & -1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{-r_2 + r_3 \to r_3} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{array}\right] \xrightarrow[-2r_3 + r_2 \to r_2]{-r_3 + r_1 \to r_1} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 2 & 1 & -1 \\ 0 & 1 & 0 & 2 & 3 & -2 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{array}\right]$$

$$\xrightarrow{-r_2 + r_1 \to r_1} \left[\underbrace{\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}}_{I} \middle| \underbrace{\begin{array}{ccc} 0 & -2 & 1 \\ 2 & 3 & -2 \\ -1 & -1 & 1 \end{array}}_{A^{-1}} \right]$$

$$\Rightarrow A^{-1} = \begin{bmatrix} 0 & -2 & 1 \\ 2 & 3 & -2 \\ -1 & -1 & 1 \end{bmatrix} \quad //$$

Question 2. [25 pts] Let $f(x,y,z) = x^4y + y^2z^3$.

(a) [10 pts] Find the first partial derivatives, f_x, f_y and f_z, of f.

$$f_x = 4x^3y$$

$$f_y = x^4 + 2yz^3$$

$$f_z = 3y^2z^2$$

(b) [15 pts] If $x = ue^v$, $y = uv$ and $z = u+v$, find $\dfrac{\partial f}{\partial u}$ and $\dfrac{\partial f}{\partial v}$ in terms of u and v (do not simplify).

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial u}$$

$$= 4x^3y.e^v + (x^4+2yz^3).v + 3y^2z^2.1$$

$$= 4.(ue^v)^3uve^v + [(ue^v)^4 + 2uv(u+v)^3]v$$

$$+ 3(uv)^2(u+v)^2$$

$$\frac{\partial f}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial v}$$

$$= 4x^3y.ue^v + (x^4+2yz^3).u + 3y^2z^2.1$$

$$= 4(ue^v)^3.uv.ue^v + [(u.e^v)^4 + 2uv(u+v)^3].u$$

$$+ 3(uv)^2(u+v)^2$$

Question 3. [25 pts] Let $f(x, y) = x^4 + y^4 - 4xy + 1$.

(a) [10 pts] Find all critical points of f.

$f_x = 4x^3 - 4y = 0 \Rightarrow y = x^3$

$f_y = 4y^3 - 4x = 0 \Rightarrow x = y^3$

$x = (x^3)^3 \Rightarrow x = x^9 \Rightarrow x^9 - x = 0 \Rightarrow$

$x \cdot (x^8 - 1) = 0 \Rightarrow x = 0, \; x = 1$ and $x = -1$.

Thus, three cricital points are: $\boxed{(0,0),\ (1,1) \text{ and } (-1,-1)}$

(b) [15 pts] Classify the critical points of f (as local maximum, local minimum or saddle point).

Second Derivatives Test: $D(x,y) = f_{xx}(x,y) \cdot f_{yy}(x,y) - [f_{xy}(x,y)]^2$

$f_{xx} = 12x^2$

$f_{yy} = 12y^2$

$f_{xy} = -4$

① $(0,0) \Rightarrow D(0,0) = f_{xx}(0,0) \cdot f_{yy}(0,0) - [f_{xy}(0,0)]^2$

$= 0 \cdot 0 - (-4)^2 = -16 < 0$

$\Rightarrow$ $\boxed{(0,0) \text{ is a saddle point}}$

② $(1,1) \Rightarrow D(1,1) = 12 \cdot 12 - (-4)^2 = 128 > 0$ and

$f_{xx}(1,1) = 12 > 0 \Rightarrow$ $\boxed{(1,1) \text{ is a local minimum point.}}$

③ $(-1,-1) \Rightarrow D(-1,-1) = 128 > 0$ and $f_{xx}(-1,-1) = 12 > 0$

$\Rightarrow$ $\boxed{(-1,-1) \text{ is a local minimum point.}}$