Question 1. [25 pts]

(a) [10 pts] The matrix A is invertible where

$$A^{-1} = \begin{bmatrix} 2 & -1 \\ 4 & 3 \end{bmatrix}.$$

Find the solution x of the system $Ax = b$, where $b = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

$Ax = b$ and A is invertible $\Rightarrow$ $x = A^{-1} b$

$$x = \begin{bmatrix} 2 & -1 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 17 \end{bmatrix} \quad //$$

(b) [15 pts] A car company sells 1000 luxury cars, 3000 medium-priced cars and 2000 compact cars per year. The company sells the luxury cars for 5000 dollars each, the medium-priced cars for 3000 dollars each, and the compact cars for 2000 dollars each.

(i) [5 pts] Represent the number of cars of each type sold in one year by a **row vector** N, and the price of each type by a **column vector** P.

$$N = \begin{bmatrix} 1000 & 3000 & 2000 \end{bmatrix} \overset{\text{OR}}{=} 10^3 \begin{bmatrix} 1 & 3 & 2 \end{bmatrix}$$

$$P = \begin{bmatrix} 5000 \\ 3000 \\ 2000 \end{bmatrix} \overset{\text{OR}}{=} 10^3 \cdot \begin{bmatrix} 5 \\ 3 \\ 2 \end{bmatrix}$$

(ii) [10 pts] Use N and P obitaned in (i) to find the total amount of money received by the company from the sale of these cars.

Total money $= N.P$

$$= 10^3 \begin{bmatrix} 1 & 3 & 2 \end{bmatrix} 10^3 \begin{bmatrix} 5 \\ 3 \\ 2 \end{bmatrix}$$

$$= 10^6 \begin{bmatrix} 1 & 3 & 2 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \\ 2 \end{bmatrix} = 10^6 . 18$$

$$= 18{,}000{,}000 \quad //$$

Question 2. [25 pts]

(a) [10 pts] Evaluate $\int_1^2 \int_0^1 x^2 y \, dy \, dx$.

$$\int_1^2 \int_0^1 x^2 y \, dy \, dx = \int_1^2 \left[\frac{x^2 y^2}{2} \Big|_{y=0}^{y=1} \right] dx$$

$$= \int_1^2 \left[\frac{x^2}{2} - 0 \right] dx$$

$$= \int_1^2 \frac{x^2}{2} \, dx$$

$$= \frac{x^3}{6} \Big|_{x=1}^{x=2} = \frac{8}{6} - \frac{1}{6}$$

$$= \frac{7}{6} \quad //$$

(b) [15 pts] Evaluate $\int_1^{\ln 8} \int_0^{\ln y} e^{x+y} \, dx \, dy$.

$$\int_1^{\ln 8} \int_0^{\ln y} e^{x+y} \, dx \, dy = \int_1^{\ln 8} \left[e^{x+y} \Big|_{x=0}^{x=\ln y} \right] dy$$

$$= \int_1^{\ln 8} \left[y e^y - e^y \right] dy \quad (*)$$

$$= y e^y - 2 e^y \Big|_{y=1}^{y=\ln 8}$$

$$= (\ln 8 . e^{\ln 8} - 2 e^{\ln 8}) - (e - 2e)$$

$$= 8 \ln 8 - 16 + e \quad //$$

(*)

$\int y e^y dy = I \Rightarrow$

$u = y$, $e^y dy = dv$

$du = dy$, $e^y = v$

$I = u.v - \int v du$

$= y e^y - \int e^y dy$

$= y e^y - e^y$

Question 3. [25 pts] Use the method of Lagrange multipliers to find the minimum and maximum values of $f(x, y) = 4x + 6y$, subject to $x^2 + y^2 = 13$.

$f(x,y) = 4x + 6y$ and $g(x,y) = x^2 + y^2 - 13$

Method of Lagrange multipliers:

$F(x,y,\lambda) = f(x,y) - \lambda . g(x,y)$

$F(x,y,\lambda) = 4x + 6y - \lambda . (x^2 + y^2 - 13)$

$F_x = 4 - 2\lambda x = 0 \Rightarrow x = \frac{2}{\lambda}$

$F_y = 6 - 2\lambda y = 0 \Rightarrow y = \frac{3}{\lambda}$

$F_\lambda = -(x^2 + y^2 - 13) = 0$

$\Rightarrow \frac{4}{\lambda^2} + \frac{9}{\lambda^2} - 13 = 0$

$\Rightarrow \lambda^2 = 1$ So $\lambda = 1$ or $\lambda = -1$.

If $\lambda = 1$, $x = 2$ and $y = 3$ $\Rightarrow (2, 3)$

If $\lambda = -1$, $x = -2$ and $y = -3$ $\Rightarrow (-2, -3)$

$f(2, 3) = 26$ [max value of f on $x^2 + y^2 = 13$]

$f(-2, -3) = -26$ [min " " " " "]