2.2 SEPARABLE EQUATIONS

A simple class of first-order differential equations that can be solved using integration is the class of **separable equations.** These are equations

$$\frac{dy}{dx} = f(x, y) , \tag{1}$$

that can be rewritten to isolate the variables x and y (together with their differentials dx and dy) on opposite sides of the equation, as in

$$h(y)dy = g(x)dx .$$

So the original right-hand side $f(x, y)$ must have the factored form

$$f(x, y) = g(x) \cdot \frac{1}{h(y)} .$$

More formally, we write $p(y) = 1/h(y)$ and present the following definition.

Separable Equation

Definition 1. If the right-hand side of the equation

$$\frac{dy}{dx} = f(x, y)$$

can be expressed as a function $g(x)$ that depends only on x times a function $p(y)$ that depends only on y, then the differential equation is called **separable.**†

In other words, a first-order equation is separable if it can be written in the form

$$\frac{dy}{dx} = g(x)p(y) .$$

For example, the equation

$$\frac{dy}{dx} = \frac{2x + xy}{y^2 + 1}$$

is separable, since (if one is sufficiently alert to detect the factorization)

$$\frac{2x + xy}{y^2 + 1} = x\frac{2 + y}{y^2 + 1} = g(x)p(y) .$$

However, the equation

$$\frac{dy}{dx} = 1 + xy$$

admits no such factorization of the right-hand side and so is not separable.

Informally speaking, one solves separable equations by performing the separation and then integrating each side.

†*Historical Footnote:* A procedure for solving separable equations was discovered implicitly by Gottfried Leibniz in 1691. The explicit technique called separation of variables was formalized by John Bernoulli in 1694.

Method for Solving Separable Equations

To solve the equation

$$\frac{dy}{dx} = g(x)p(y) \tag{2}$$

multiply by dx and by $h(y) := 1/p(y)$ to obtain

$$h(y)dy = g(x)dx \ .$$

Then integrate both sides:

$$\int h(y)dy = \int g(x)dx \ ,$$

$$H(y) = G(x) + C \ , \tag{3}$$

where we have merged the two constants of integration into a single symbol C. The last equation gives an implicit solution to the differential equation.

Caution: Constant functions $y \equiv c$ such that $p(c) = 0$ are also solutions to (2), which may or may not be included in (3) (as we shall see in Example 3).

We will look at the mathematical justification of this "streamlined" procedure shortly, but first we study some examples.

Example 1 Solve the nonlinear equation

$$\frac{dy}{dx} = \frac{x-5}{y^2} \ .$$

Solution Following the streamlined approach, we separate the variables and rewrite the equation in the form

$$y^2\,dy = (x-5)dx \ .$$

Integrating, we have

$$\int y^2\,dy = \int (x-5)dx$$

$$\frac{y^3}{3} = \frac{x^2}{2} - 5x + C \ ,$$

and solving this last equation for y gives

$$y = \left(\frac{3x^2}{2} - 15x + 3C\right)^{1/3} .$$

Since C is a constant of integration that can be any real number, $3C$ can also be any real number. Replacing $3C$ by the single symbol K, we then have

$$y = \left(\frac{3x^2}{2} - 15x + K\right)^{1/3} .$$

If we wish to abide by the custom of letting C represent an arbitrary constant, we can go one step further and use C instead of K in the final answer. This solution family is graphed in Figure 2.3 on page 40. ◆

As Example 1 attests, separable equations are among the easiest to solve. However, the procedure does require a facility for computing integrals. Many of the procedures to be discussed in the text also require a familiarity with the techniques of integration. For this reason

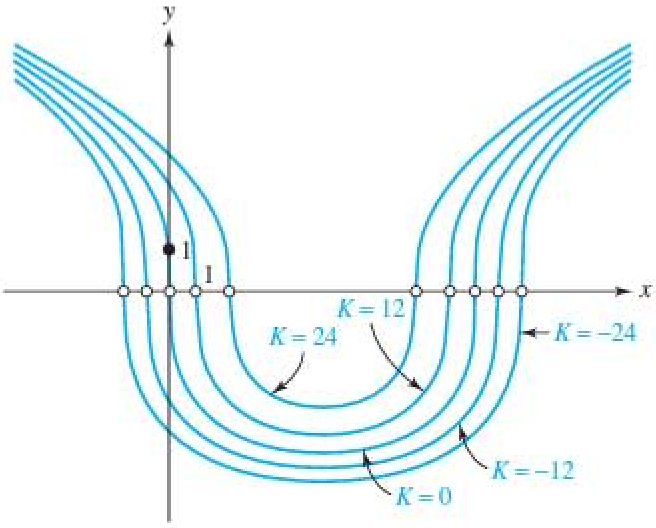

Figure 2.3 Family of solutions for Example 1[†]

we have provided a review of integration methods in Appendix A and a brief table of integrals on the inside front cover.

Example 2 Solve the initial value problem

$$\frac{dy}{dx} = \frac{y-1}{x+3}, \qquad y(-1) = 0 . \tag{4}$$

Solution Separating the variables and integrating gives

$$\frac{dy}{y-1} = \frac{dx}{x+3},$$

$$\int \frac{dy}{y-1} = \int \frac{dx}{x+3},$$

$$\ln|y-1| = \ln|x+3| + C . \tag{5}$$

At this point, we can either solve for y explicitly (retaining the constant C) or use the initial condition to determine C and then solve explicitly for y. Let's try the first approach.

Exponentiating equation (5), we have

$$e^{\ln|y-1|} = e^{\ln|x+3|+C} = e^C e^{\ln|x+3|},$$

$$|y-1| = e^C|x+3| = C_1|x+3| , \tag{6}$$

where $C_1 := e^C$.[††] Now, depending on the values of y, we have $|y-1| = \pm(y-1)$; and similarly, $|x+3| = \pm(x+3)$. Thus, (6) can be written as

$$y - 1 = \pm C_1(x+3) \quad \text{or} \quad y = 1 \pm C_1(x+3) ,$$

[†]The gaps in the curves reflect the fact that in the original differential equation, y appears in the denominator, so that y = 0 must be excluded.

[††]Recall that the symbol := means "is defined to be."

where the choice of sign depends (as we said) on the values of x and y. Because C_1 is a *positive* constant (recall that $C_1 = e^C > 0$), we can replace $\pm C_1$ by K, where K now represents an *arbitrary* nonzero constant. We then obtain

(7) $$y = 1 + K(x + 3) .$$

Finally, we determine K such that the initial condition $y(-1) = 0$ is satisfied. Putting $x = -1$ and $y = 0$ in equation (7) gives

$$0 = 1 + K(-1 + 3) = 1 + 2K ,$$

and so $K = -1/2$. Thus, the solution to the initial value problem is

(8) $$y = 1 - \frac{1}{2}(x + 3) = -\frac{1}{2}(x + 1) .$$

Alternative Approach. The second approach is to first set $x = -1$ and $y = 0$ in equation (5) and solve for C. In this case, we obtain

$$\ln|0 - 1| = \ln|-1 + 3| + C ,$$
$$0 = \ln 1 = \ln 2 + C ,$$

and so $C = -\ln 2$. Thus, from (5), the solution y is given implicitly by

$$\ln(1 - y) = \ln(x + 3) - \ln 2 .$$

Here we have replaced $|y - 1|$ by $1 - y$ and $|x + 3|$ by $x + 3$, since we are interested in x and y near the initial values $x = -1$, $y = 0$ (for such values, $y - 1 < 0$ and $x + 3 > 0$). Solving for y, we find

$$\ln(1 - y) = \ln(x + 3) - \ln 2 = \ln\left(\frac{x + 3}{2}\right) ,$$

$$1 - y = \frac{x + 3}{2} ,$$

$$y = 1 - \frac{1}{2}(x + 3) = -\frac{1}{2}(x + 1) ,$$

which agrees with the solution (8) found by the first method. ◆

Example 3 Solve the nonlinear equation

(9) $$\frac{dy}{dx} = \frac{6x^5 - 2x + 1}{\cos y + e^y} .$$

Solution Separating variables and integrating, we find

$$(\cos y + e^y)dy = (6x^5 - 2x + 1)dx ,$$

$$\int(\cos y + e^y)dy = \int(6x^5 - 2x + 1)dx ,$$

$$\sin y + e^y = x^6 - x^2 + x + C .$$

At this point, we reach an impasse. We would like to solve for y explicitly, but we cannot. This is often the case in solving nonlinear first-order equations. Consequently, when we say "solve the equation," we must on occasion be content if only an implicit form of the solution has been found. ◆

The separation of variables technique, as well as several other techniques discussed in this book, entails rewriting a differential equation by performing certain algebraic operations on it.

2.2 EXERCISES

In Problems 1–6, determine whether the given differential equation is separable.

1. $\frac{dy}{dx} - \sin(x + y) = 0$
2. $\frac{dy}{dx} = 4y^2 - 3y + 1$
3. $\frac{ds}{dt} = t\ln(s^{2t}) + 8t^2$
4. $\frac{dy}{dx} = \frac{ye^{x+y}}{x^2 + 2}$
5. $(xy^2 + 3y^2)dy - 2x\,dx = 0$
6. $s^2 + \frac{ds}{dt} = \frac{s + 1}{st}$

In Problems 7–16, solve the equation.

7. $\frac{dx}{dt} = 3xt^2$
8. $x\frac{dy}{dx} = \frac{1}{y^3}$
9. $\frac{dy}{dx} = \frac{x}{y^2\sqrt{1 + x}}$
10. $\frac{dx}{dt} = \frac{t}{xe^{t+2x}}$
11. $\frac{dy}{dx} = \frac{\sec^2 y}{1 + x^2}$
12. $x\frac{dv}{dx} = \frac{1 - 4v^2}{3v}$
13. $\frac{dx}{dt} - x^3 = x$
14. $\frac{dy}{dx} = 3x^2(1 + y^2)^{3/2}$
15. $y^{-1}\,dy + ye^{\cos x}\sin x\,dx = 0$
16. $(x + xy^2)dx + e^{x^2}y\,dy = 0$

In Problems 17–26, solve the initial value problem.

17. $y' = x^3(1 - y)$, $y(0) = 3$
18. $\frac{dy}{dx} = (1 + y^2)\tan x$, $y(0) = \sqrt{3}$
19. $\frac{1}{2}\frac{dy}{dx} = \sqrt{y + 1}\cos x$, $y(\pi) = 0$
20. $x^2\frac{dy}{dx} = \frac{4x^2 - x - 2}{(x + 1)(y + 1)}$, $y(1) = 1$
21. $\frac{1}{\theta}\frac{dy}{d\theta} = \frac{y\sin\theta}{y^2 + 1}$, $y(\pi) = 1$
22. $x^2\,dx + 2y\,dy = 0$, $y(0) = 2$
23. $t^{-1}\frac{dy}{dt} = 2\cos^2 y$, $y(0) = \pi/4$
24. $\frac{dy}{dx} = 8x^3e^{-2y}$, $y(1) = 0$
25. $\frac{dy}{dx} = x^2(1 + y)$, $y(0) = 3$
26. $\sqrt{y}dx + (1 + x)dy = 0$, $y(0) = 1$

27. **Solutions Not Expressible in Terms of Elementary Functions.** As discussed in calculus, certain indefinite integrals (antiderivatives) such as $\int e^{x^2}dx$ cannot be expressed in finite terms using elementary functions. When such an integral is encountered while solving a differential equation, it is often helpful to use definite integration (integrals with variable upper limit). For example, consider the initial value problem

$$\frac{dy}{dx} = e^{x^2}y^2 , \qquad y(2) = 1 .$$

The differential equation separates if we divide by y^2 and multiply by dx. We integrate the separated equation from $x = 2$ to $x = x_1$ and find

$$\int_{x=2}^{x=x_1} e^{x^2}\,dx = \int_{x=2}^{x=x_1} \frac{dy}{y^2} = -\frac{1}{y}\bigg|_{x=2}^{x=x_1} = -\frac{1}{y(x_1)} + \frac{1}{y(2)} .$$

If we let t be the variable of integration and replace x_1 by x and $y(2)$ by 1, then we can express the solution to the initial value problem by

$$y(x) = \left(1 - \int_2^x e^{t^2}dt\right)^{-1} .$$

Use definite integration to find an explicit solution to the initial value problems in parts (a)–(c).

(a) $dy/dx = e^{x^2}$, $y(0) = 0$
(b) $dy/dx = e^{x^2}y^{-2}$, $y(0) = 1$
(c) $dy/dx = \sqrt{1 + \sin x}\,(1 + y^2)$, $y(0) = 1$
(d) Use a numerical integration algorithm (such as Simpson's rule, described in Appendix C) to approximate the solution to part (b) at $x = 0.5$ to three decimal places.

28. Sketch the solution to the initial value problem

$$\frac{dy}{dt} = 2y - 2yt , \qquad y(0) = 3$$

and determine its maximum value.

29. **Uniqueness Questions.** In Chapter 1 we indicated that in applications most *initial value problems* will have a unique solution. In fact, the existence of unique

2.3 LINEAR EQUATIONS

A type of first-order differential equation that occurs frequently in applications is the linear equation. Recall from Section 1.1 that a **linear first-order equation** is an equation that can be expressed in the form

$$(1)\qquad a_1(x)\frac{dy}{dx} + a_0(x)y = b(x) \;,$$

where $a_1(x)$, $a_0(x)$, and $b(x)$ depend only on the independent variable x, not on y.

For example, the equation

$$x^2\sin x - (\cos x)y = (\sin x)\frac{dy}{dx}$$

is linear, because it can be rewritten in the form

$$(\sin x)\frac{dy}{dx} + (\cos x)y = x^2\sin x \;.$$

However, the equation

$$y\frac{dy}{dx} + (\sin x)y^3 = e^x + 1$$

is not linear; it cannot be put in the form of equation (1) due to the presence of the y^3 and $y\,dy/dx$ terms.

There are two situations for which the solution of a linear differential equation is quite immediate. The first arises if the coefficient $a_0(x)$ is identically zero, for then equation (1) reduces to

$$(2)\qquad a_1(x)\frac{dy}{dx} = b(x) \;,$$

which is equivalent to

$$y(x) = \int \frac{b(x)}{a_1(x)}dx + C$$

[as long as $a_1(x)$ is not zero].

The second is less trivial. Note that if $a_0(x)$ happens to equal the derivative of $a_1(x)$—that is, $a_0(x) = a_1'(x)$—then the two terms on the left-hand side of equation (1) simply comprise the derivative of the product $a_1(x)y$:

$$a_1(x)y' + a_0(x)y = a_1(x)y' + a_1'(x)y = \frac{d}{dx}[a_1(x)y] \;.$$

Therefore equation (1) becomes

$$(3)\qquad \frac{d}{dx}[a_1(x)y] = b(x)$$

and the solution is again elementary:

$$a_1(x)y = \int b(x)dx + C \ ,$$

$$y(x) = \frac{1}{a_1(x)}\left[\int b(x)dx + C\right] \ .$$

One can seldom rewrite a linear differential equation so that it reduces to a form as simple as (2). However, the form (3) can be achieved through multiplication of the original equation (1) by a well-chosen function $\mu(x)$. Such a function $\mu(x)$ is then called an "integrating factor" for equation (1). The easiest way to see this is first to divide the original equation (1) by $a_1(x)$ and put it into **standard form**

(4) $$\frac{dy}{dx} + P(x)y = Q(x) \ ,$$

where $P(x) = a_0(x)/a_1(x)$ and $Q(x) = b(x)/a_1(x)$.

Next we wish to determine $\mu(x)$ so that the left-hand side of the multiplied equation

(5) $$\mu(x)\frac{dy}{dx} + \mu(x)P(x)y = \mu(x)Q(x)$$

is just the derivative of the product $\mu(x)y$:

$$\mu(x)\frac{dy}{dx} + \mu(x)P(x)y = \frac{d}{dx}[\mu(x)y] = \mu(x)\frac{dy}{dx} + \mu'(x)y \ .$$

Clearly, this requires that μ satisfy

(6) $$\mu' = \mu P \ .$$

To find such a function, we recognize that equation (6) is a separable differential equation, which we can write as $(1/\mu)\,d\mu = P(x)dx$. Integrating both sides gives

(7) $$\mu(x) = e^{\int P(x)dx} \ .$$

With this choice† for $\mu(x)$, equation (5) becomes

$$\frac{d}{dx}[\mu(x)y] = \mu(x)Q(x) \ ,$$

which has the solution

(8) $$y(x) = \frac{1}{\mu(x)}\left[\int \mu(x)Q(x)dx + C\right] \ .$$

Here C is an arbitrary constant, so (8) gives a one-parameter family of solutions to (4). This form is known as the **general solution** to (4).

---

†Any choice of the integration constant in $\int P(x)dx$ will produce a suitable $\mu(x)$.

We can summarize the method for solving linear equations as follows.

Method for Solving Linear Equations

(a) Write the equation in the standard form

$$\frac{dy}{dx} + P(x)y = Q(x) \ .$$

(b) Calculate the integrating factor $\mu(x)$ by the formula

$$\mu(x) = \exp\left[\int P(x)\,dx\right] \ .$$

(c) Multiply the equation in standard form by $\mu(x)$ and, recalling that the left-hand side is just $\frac{d}{dx}\left[\mu(x)y\right]$, obtain

$$\underbrace{\mu(x)\frac{dy}{dx} + P(x)\mu(x)y}_{\frac{d}{dx}\left[\mu(x)y\right]} = \mu(x)Q(x) \ ,$$
$$\frac{d}{dx}\left[\mu(x)y\right] = \mu(x)Q(x) \ .$$

(d) Integrate the last equation and solve for y by dividing by $\mu(x)$ to obtain (8).

Example 1 Find the general solution to

$$\frac{1}{x}\frac{dy}{dx} - \frac{2y}{x^2} = x\cos x \ , \qquad x > 0 \ . \tag{9}$$

Solution To put this linear equation in standard form, we multiply by x to obtain

$$\frac{dy}{dx} - \frac{2}{x}y = x^2\cos x \ . \tag{10}$$

Here $P(x) = -2/x$, so

$$\int P(x)dx = \int \frac{-2}{x}dx = -2\ln|x| \ .$$

Thus, an integrating factor is

$$\mu(x) = e^{-2\ln|x|} = e^{\ln(x^{-2})} = x^{-2} \ .$$

Multiplying equation (10) by $\mu(x)$ yields

$$\underbrace{x^{-2}\frac{dy}{dx} - 2x^{-3}y}_{\frac{d}{dx}(x^{-2}y)} = \cos x \ ,$$
$$\frac{d}{dx}(x^{-2}y) = \cos x \ .$$

We now integrate both sides and solve for y to find

$$x^{-2}y = \int \cos x \, dx = \sin x + C$$

$$y = x^2 \sin x + Cx^2 \ . \tag{11}$$

It is easily checked that this solution is valid for all $x > 0$. In Figure 2.5 we have sketched solutions for various values of the constant C in (11). ◆

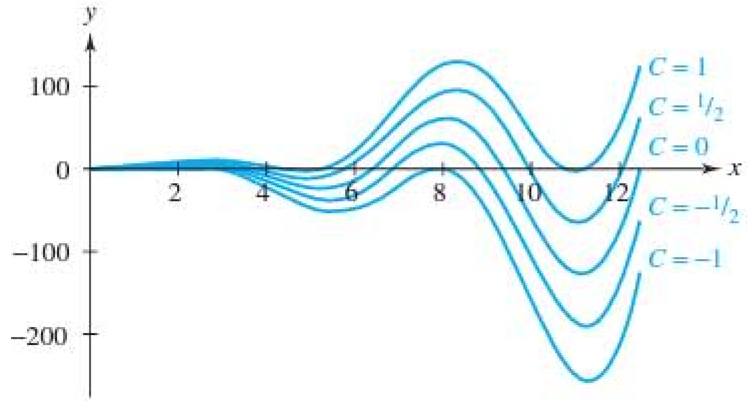

Figure 2.5 Graph of $y = x^2 \sin x + Cx^2$ for five values of the constant C

In the next example, we encounter a linear equation that arises in the study of the radioactive decay of an isotope.

Example 2 A rock contains two radioactive isotopes, RA_1 and RA_2, that belong to the same radioactive series; that is, RA_1 decays into RA_2, which then decays into stable atoms. Assume that the rate at which RA_1 decays into RA_2 is $50e^{-10t}$ kg/sec. Because the rate of decay of RA_2 is proportional to the mass $y(t)$ of RA_2 present, the rate of change in RA_2 is

$$\frac{dy}{dt} = \text{rate of creation} - \text{rate of decay} \ ,$$

$$\frac{dy}{dt} = 50e^{-10t} - ky \ , \tag{12}$$

where $k > 0$ is the decay constant. If $k = 2$/sec and initially $y(0) = 40$ kg, find the mass $y(t)$ of RA_2 for $t \geq 0$.

Solution Equation (12) is linear, so we begin by writing it in standard form

$$\frac{dy}{dt} + 2y = 50e^{-10t} \ , \qquad y(0) = 40 \ , \tag{13}$$

where we have substituted $k = 2$ and displayed the initial condition. We now see that $P(t) = 2$, so $\int P(t)dt = \int 2\, dt = 2t$. Thus, an integrating factor is $\mu(t) = e^{2t}$. Multiplying equation (13) by $\mu(t)$ yields

$$\underbrace{e^{2t}\frac{dy}{dt} + 2e^{2t}y}_{\frac{d}{dt}(e^{2t}y)} = 50e^{-10t+2t} = 50e^{-8t} \ ,$$

$$\frac{d}{dt}(e^{2t}y) = 50e^{-8t} \ .$$

Integrating both sides and solving for y, we find

$$e^{2t}y = -\frac{25}{4}e^{-8t} + C ,$$

$$y = -\frac{25}{4}e^{-10t} + Ce^{-2t} .$$

Substituting $t = 0$ and $y(0) = 40$ gives

$$40 = -\frac{25}{4}e^{0} + Ce^{0} = -\frac{25}{4} + C ,$$

so $C = 40 + 25/4 = 185/4$. Thus, the mass $y(t)$ of RA_2 at time t is given by

(14) $$y(t) = \left(\frac{185}{4}\right)e^{-2t} - \left(\frac{25}{4}\right)e^{-10t} , \qquad t \geq 0 .$$ ◆

Example 3 For the initial value problem

$$y' + y = \sqrt{1 + \cos^2 x} , \qquad y(1) = 4 ,$$

find the value of $y(2)$.

Solution The integrating factor for the differential equation is, from equation (7),

$$\mu(x) = e^{\int 1\, dx} = e^{x} .$$

The general solution form (8) thus reads

$$y(x) = e^{-x}\left(\int e^{x}\sqrt{1 + \cos^2 x}\, dx + C\right) .$$

However, this indefinite integral cannot be expressed in finite terms with elementary functions (recall a similar situation in Problem 27 of Exercises 2.2). Because we *can* use numerical algorithms such as Simpson's rule (Appendix C) to perform *definite* integration, we revert to the form (5), which in this case reads

$$\frac{d}{dx}(e^{x}y) = e^{x}\sqrt{1 + \cos^2 x} ,$$

and take the definite integral from the initial value $x = 1$ to the desired value $x = 2$:

$$e^{x}y\Big|_{x=1}^{x=2} = e^{2}y(2) - e^{1}y(1) = \int_{x=1}^{x=2} e^{x}\sqrt{1 + \cos^2 x}\, dx .$$

Inserting the given value of $y(1)$ and solving, we express

$$y(2) = e^{-2+1}(4) + e^{-2}\int_{1}^{2} e^{x}\sqrt{1 + \cos^2 x}\, dx .$$

Using Simpson's rule, we find that the definite integral is approximately 4.841, so

$$y(2) \approx 4e^{-1} + 4.841e^{-2} \approx 2.127 .$$ ◆

In Example 3 we had no difficulty expressing the integral for the integrating factor $\mu(x) = e^{\int 1\,dx} = e^x$. Clearly, situations will arise where this integral, too, cannot be expressed with elementary functions. In such cases we must again resort to a numerical procedure such as Euler's method (Section 1.4) or to a "nested loop" implementation of Simpson's rule. You are invited to explore such a possibility in Problem 27.

Because we have established explicit formulas for the solutions to *linear* first-order differential equations, we get as a dividend a direct proof of the following theorem.

Existence and Uniqueness of Solution

Theorem 1. Suppose $P(x)$ and $Q(x)$ are continuous on an interval (a, b) that contains the point x_0. Then for any choice of initial value y_0, there exists a unique solution $y(x)$ on (a, b) to the initial value problem

$$\frac{dy}{dx} + P(x)y = Q(x)\,, \qquad y(x_0) = y_0\,. \tag{15}$$

In fact, the solution is given by (8) for a suitable value of C.

The essentials of the proof of Theorem 1 are contained in the deliberations leading to equation (8); Problem 34 provides the details. This theorem differs from Theorem 1 on page 11 in that for the *linear* initial value problem (15), we have the existence and uniqueness of the solution on the *whole* interval (a, b), rather than on some smaller unspecified interval about x_0.

The theory of linear differential equations is an important branch of mathematics not only because these equations occur in applications but also because of the elegant structure associated with them. For example, first-order linear equations always have a general solution given by equation (8). Some further properties of first-order linear equations are described in Problems 28 and 36. Higher-order linear equations are treated in Chapters 4, 6, and 8.

2.3 EXERCISES

In Problems 1–6, determine whether the given equation is separable, linear, neither, or both.

1. $\frac{dx}{dt} + xt = e^x$

2. $x^2\frac{dy}{dx} + \sin x - y = 0$

3. $3t = e^t\frac{dy}{dt} + y\ln t$

4. $(t^2 + 1)\frac{dy}{dt} = yt - y$

5. $3r = \frac{dr}{d\theta} - \theta^3$

6. $x\frac{dx}{dt} + t^2x = \sin t$

In Problems 7–16, obtain the general solution to the equation.

7. $\frac{dy}{dx} = \frac{y}{x} + 2x + 1$

8. $\frac{dy}{dx} - y - e^{3x} = 0$

9. $x\frac{dy}{dx} + 2y = x^{-3}$

10. $\frac{dr}{d\theta} + r\tan\theta = \sec\theta$

11. $(t + y + 1)dt - dy = 0$

12. $\frac{dy}{dx} = x^2e^{-4x} - 4y$

13. $y\frac{dx}{dy} + 2x = 5y^3$

14. $x\frac{dy}{dx} + 3(y + x^2) = \frac{\sin x}{x}$

15. $(x^2 + 1)\frac{dy}{dx} + xy - x = 0$

16. $(1 - x^2)\frac{dy}{dx} - x^2y = (1 + x)\sqrt{1 - x^2}$

In Problems 17–22, solve the initial value problem.

17. $\frac{dy}{dx} - \frac{y}{x} = xe^x , \quad y(1) = e - 1$

18. $\frac{dy}{dx} + 4y - e^{-x} = 0 , \quad y(0) = \frac{4}{3}$

19. $t^2\frac{dx}{dt} + 3tx = t^4 \ln t + 1, \quad x(1) = 0$

20. $\frac{dy}{dx} + \frac{3y}{x} + 2 = 3x , \quad y(1) = 1$

21. $\cos x\frac{dy}{dx} + y \sin x = 2x \cos^2 x ,$

$y\left(\frac{\pi}{4}\right) = \frac{-15\sqrt{2}\pi^2}{32}$

22. $\sin x\frac{dy}{dx} + y \cos x = x \sin x , \quad y\left(\frac{\pi}{2}\right) = 2$

23. Radioactive Decay. In Example 2 assume that the rate at which RA_1 decays into RA_2 is $40e^{-20t}$ kg/sec and the decay constant for RA_2 is $k = 5$/sec. Find the mass $y(t)$ of RA_2 for $t \geq 0$ if initially $y(0) = 10$ kg.

24. In Example 2 the decay constant for isotope RA_1 was 10/sec, which expresses itself in the exponent of the rate term $50e^{-10t}$ kg/sec. When the decay constant for RA_2 is $k = 2$/sec, we see that in formula (14) for y the term $(185/4)e^{-2t}$ eventually dominates (has greater magnitude for t large).

(a) Redo Example 2 taking $k = 20$/sec. Now which term in the solution eventually dominates?

(b) Redo Example 2 taking $k = 10$/sec.

25. (a) Using definite integration, show that the solution to the initial value problem

$$\frac{dy}{dx} + 2xy = 1 , \quad y(2) = 1 ,$$

can be expressed as

$$y(x) = e^{-x^2}\left(e^4 + \int_2^x e^{t^2}\,dt\right) .$$

(b) Use numerical integration (such as Simpson's rule, Appendix C) to approximate the solution at $x = 3$.

26. Use numerical integration (such as Simpson's rule, Appendix C) to approximate the solution, at $x = 1$, to the initial value problem

$$\frac{dy}{dx} + \frac{\sin 2x}{2(1 + \sin^2 x)}y = 1 , \quad y(0) = 0 .$$

Ensure your approximation is accurate to three decimal places.

27. Consider the initial value problem

$$\frac{dy}{dx} + \sqrt{1 + \sin^2 x}\, y = x , \quad y(0) = 2 .$$

(a) Using definite integration, show that the integrating factor for the differential equation can be written as

$$\mu(x) = \exp\left(\int_0^x \sqrt{1 + \sin^2 t}\, dt\right)$$

and that the solution to the initial value problem is

$$y(x) = \frac{1}{\mu(x)}\int_0^x \mu(s)\, s\, ds + \frac{2}{\mu(x)} .$$

(b) Obtain an approximation to the solution at $x = 1$ by using numerical integration (such as Simpson's rule, Appendix C) in a nested loop to estimate values of $\mu(x)$ and, thereby, the value of

$$\int_0^1 \mu(s)\, s\, ds .$$

[*Hint:* First, use Simpson's rule to approximate $\mu(x)$ at $x = 0.1, 0.2, \ldots, 1$. Then use these values and apply Simpson's rule again to approximate $\int_0^1 \mu(s)\, s\, ds$.]

(c) Use Euler's method (Section 1.4) to approximate the solution at $x = 1$, with step sizes $h = 0.1$ and 0.05.

[A direct comparison of the merits of the two numerical schemes in parts (b) and (c) is very complicated, since it should take into account the number of functional evaluations in each algorithm as well as the inherent accuracies.]

28. Constant Multiples of Solutions.

(a) Show that $y = e^{-x}$ is a solution of the linear equation

$$\frac{dy}{dx} + y = 0 , \tag{16}$$

and $y = x^{-1}$ is a solution of the nonlinear equation

$$\frac{dy}{dx} + y^2 = 0 . \tag{17}$$

(b) Show that for any constant C, the function Ce^{-x} is a solution of equation (16), while Cx^{-1} is a solution of equation (17) only when $C = 0$ or 1.

(c) Show that for any linear equation of the form

$$\frac{dy}{dx} + P(x)y = 0 ,$$

if $\hat{y}(x)$ is a solution, then for any constant C the function $C\hat{y}(x)$ is also a solution.

38. Use the separation of variables technique to derive the solution (7) to the differential equation (6).

39. The temperature T (in units of 100°F) of a university classroom on a cold winter day varies with time t (in hours) as

$$\frac{dT}{dt} = \begin{cases} 1 - T\,, & \text{if heating unit is ON.} \\ -T\,, & \text{if heating unit is OFF.} \end{cases}$$

Suppose $T = 0$ at 9:00 A.M., the heating unit is ON from 9–10 A.M., OFF from 10–11 A.M., ON again from 11 A.M.–noon, and so on for the rest of the day. How warm will the classroom be at noon? At 5:00 P.M.?

2.4 EXACT EQUATIONS

Suppose the mathematical function $F(x,y)$ represents some physical quantity, such as temperature, in a region of the xy-plane. Then the level curves of F, where $F(x,y) = constant$, could be interpreted as isotherms on a weather map, as depicted in Figure 2.8.

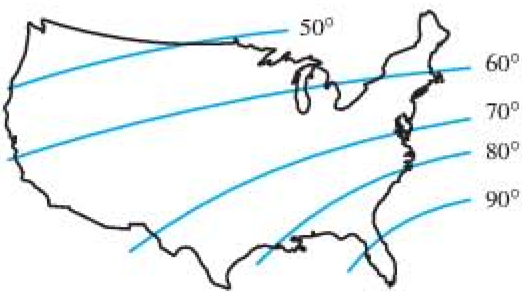

Figure 2.8 Level curves of $F(x, y)$

How does one calculate the slope of the tangent to a level curve? It is accomplished by implicit differentiation: One takes the derivative, with respect to x, of both sides of the equation $F(x,y) = C$, taking into account that y depends on x along the curve:

$$\frac{d}{dx}F(x,y) = \frac{d}{dx}(C) \quad \text{or}$$

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0\,, \tag{1}$$

and solves for the slope:

$$\frac{dy}{dx} = f(x,y) = -\frac{\partial F/\partial x}{\partial F/\partial y}\,. \tag{2}$$

The expression obtained by formally multiplying the left-hand member of (1) by dx is known as the *total differential* of F, written dF:

$$dF := \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy\,,$$

and our procedure for obtaining the equation for the slope $f(x,y)$ of the level curve $F(x,y) = C$ can be expressed as setting the total differential $dF = 0$ and solving.

Because equation (2) has the form of a differential equation, we should be able to reverse this logic and come up with a very easy technique for solving some differential equations. After all, any first-order differential equation $dy/dx = f(x, y)$ can be rewritten in the (differential) form

$$M(x, y)dx + N(x, y)dy = 0 \tag{3}$$

(in a variety of ways). Now, if the left-hand side of equation (3) can be identified as a total differential,

$$M(x, y)dx + N(x, y)dy = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy = dF(x, y) ,$$

then its solutions are given (implicitly) by the level curves

$$F(x, y) = C$$

for an arbitrary constant C.

Example 1 Solve the differential equation

$$\frac{dy}{dx} = -\frac{2xy^2 + 1}{2x^2y} .$$

Solution Some of the choices of differential forms corresponding to this equation are

$$(2xy^2 + 1)dx + 2x^2y\,dy = 0 ,$$

$$\frac{2xy^2 + 1}{2x^2y}dx + dy = 0 ,$$

$$dx + \frac{2x^2y}{2xy^2 + 1}dy = 0 , \quad \text{etc.}$$

However, the first form is best for our purposes because it is a total differential of the function $F(x, y) = x^2y^2 + x$:

$$\begin{aligned}(2xy^2 + 1)dx + 2x^2y\,dy &= d[x^2y^2 + x] \\ &= \frac{\partial}{\partial x}(x^2y^2 + x)dx + \frac{\partial}{\partial y}(x^2y^2 + x)dy .\end{aligned}$$

Thus, the solutions are given implicitly by the formula $x^2y^2 + x = C$. See Figure 2.9. ◆

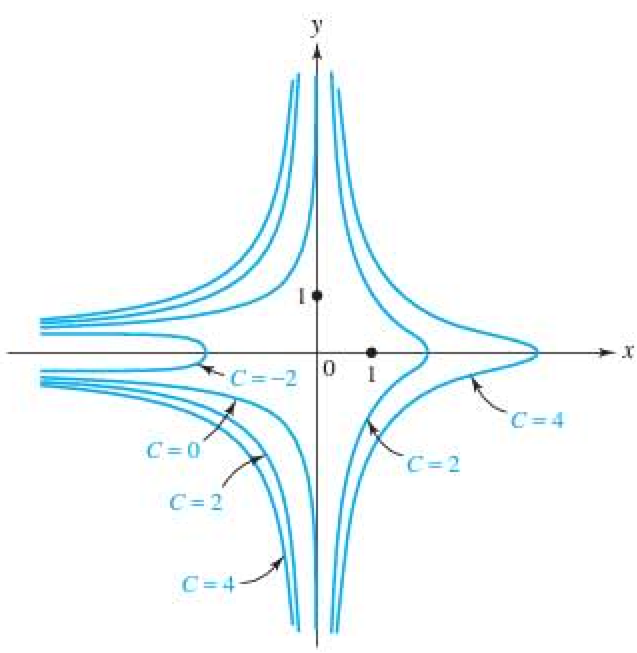

Figure 2.9 Solutions of Example 1

Next we introduce some terminology.

Exact Differential Form

Definition 2. The differential form $M(x, y)dx + N(x, y)dy$ is said to be **exact** in a rectangle R if there is a function $F(x, y)$ such that

(4) $$\frac{\partial F}{\partial x}(x, y) = M(x, y) \quad \text{and} \quad \frac{\partial F}{\partial y}(x, y) = N(x, y)$$

for all (x, y) in R. That is, the total differential of $F(x, y)$ satisfies

$$dF(x, y) = M(x, y)dx + N(x, y)dy \ .$$

If $M(x, y)dx + N(x, y)dy$ is an exact differential form, then the equation

$$M(x, y)dx + N(x, y)dy = 0$$

is called an **exact equation.**

As you might suspect, in applications a differential equation is rarely given to us in exact differential form. However, the solution procedure is so quick and simple for such equations that we devote this section to it. From Example 1, we see that what is needed is (i) a test to determine if a differential form $M(x, y)dx + N(x, y)dy$ is exact and, if so, (ii) a procedure for finding the function $F(x, y)$ itself.

The test for exactness arises from the following observation. If

$$M(x, y)dx + N(x, y)dy = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy \ ,$$

then the calculus theorem concerning the equality of continuous mixed partial derivatives

$$\frac{\partial}{\partial y}\frac{\partial F}{\partial x} = \frac{\partial}{\partial x}\frac{\partial F}{\partial y}$$

would dictate a "compatibility condition" on the functions M and N:

$$\frac{\partial}{\partial y}M(x, y) = \frac{\partial}{\partial x}N(x, y) \ .$$

In fact, Theorem 2 states that the compatibility condition is also *sufficient* for the differential form to be exact.

Test for Exactness

Theorem 2. Suppose the first partial derivatives of $M(x, y)$ and $N(x, y)$ are continuous in a rectangle R. Then

$$M(x, y)dx + N(x, y)dy = 0$$

is an exact equation in R if and only if the compatibility condition

(5) $$\frac{\partial M}{\partial y}(x, y) = \frac{\partial N}{\partial x}(x, y)$$

holds for all (x, y) in R.[†]

Before we address the proof of Theorem 2, note that in Example 1 the differential form that led to the total differential was

$$(2xy^2 + 1)dx + (2x^2y)dy = 0 \ .$$

[†] *Historical Footnote:* This theorem was proven by Leonhard Euler in 1734.

The compatibility conditions are easily confirmed:

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(2xy^2 + 1) = 4xy ,$$

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}(2x^2 y) = 4xy .$$

Also clear is the fact that the other differential forms considered,

$$\frac{2xy^2 + 1}{2x^2 y} dx + dy = 0 , \qquad dx + \frac{2x^2 y}{2xy^2 + 1} dy = 0 ,$$

do *not* meet the compatibility conditions.

Proof of Theorem 2. There are two parts to the theorem: Exactness implies compatibility, and compatibility implies exactness. First, we have seen that if the differential equation is exact, then the two members of equation (5) are simply the mixed second partials of a function $F(x, y)$. As such, their equality is ensured by the theorem of calculus that states that mixed second partials are equal if they are continuous. Because the hypothesis of Theorem 2 guarantees the latter condition, equation (5) is validated.

Rather than proceed directly with the proof of the second part of the theorem, let's derive a formula for a function $F(x, y)$ that satisfies $\partial F/\partial x = M$ and $\partial F/\partial y = N$. Integrating the first equation with respect to x yields

(6) $$F(x, y) = \int M(x, y) dx + g(y) .$$

Notice that instead of using C to represent the constant of integration, we have written $g(y)$. This is because y is held fixed while integrating with respect to x, and so our "constant" may well depend on y. To determine $g(y)$, we differentiate both sides of (6) with respect to y to obtain

(7) $$\frac{\partial F}{\partial y}(x, y) = \frac{\partial}{\partial y} \int M(x, y) dx + \frac{\partial}{\partial y} g(y) .$$

As g is a function of y alone, we can write $\partial g/\partial y = g'(y)$, and solving (7) for $g'(y)$ gives

$$g'(y) = \frac{\partial F}{\partial y}(x, y) - \frac{\partial}{\partial y} \int M(x, y) dx .$$

Since $\partial F/\partial y = N$, this last equation becomes

(8) $$g'(y) = N(x, y) - \frac{\partial}{\partial y} \int M(x, y) dx .$$

Notice that although the right-hand side of (8) indicates a possible dependence on x, *the appearances of this variable must cancel* because the left-hand side, $g'(y)$, depends only on y. By integrating (8), we can determine $g(y)$ up to a numerical constant, and therefore we can determine the function $F(x, y)$ up to a numerical constant from the functions $M(x, y)$ and $N(x, y)$.

To finish the proof of Theorem 2, we need to show that the condition (5) implies that $M\,dx + N\,dy = 0$ is an exact equation. This we do by actually exhibiting a function $F(x, y)$ that satisfies $\partial F/\partial x = M$ and $\partial F/\partial y = N$. Fortunately, we needn't look too far for such a function.

The discussion in the first part of the proof suggests (6) as a candidate, where $g'(y)$ is given by (8). Namely, we *define* $F(x, y)$ by

$$F(x,y) := \int_{x_0}^{x} M(t,y)\,dt + g(y) \ , \tag{9}$$

where (x_0, y_0) is a fixed point in the rectangle R and $g(y)$ is determined, up to a numerical constant, by the equation

$$g'(y) := N(x,y) - \frac{\partial}{\partial y}\int_{x_0}^{x} M(t,y)\,dt \ . \tag{10}$$

Before proceeding we must address an extremely important question concerning the definition of $F(x, y)$. That is, how can we be sure (in this portion of the proof) that $g'(y)$, as given in equation (10), is really a function of just y alone? To show that the right-hand side of (10) is independent of x (that is, that the appearances of the variable x cancel), all we need to do is show that its partial derivative with respect to x is zero. This is where condition (5) is utilized. We leave to the reader this computation and the verification that $F(x, y)$ satisfies conditions (4) (see Problems 35 and 36). ◆

The construction in the proof of Theorem 2 actually provides an explicit procedure for solving exact equations. Let's recap and look at some examples.

Method for Solving Exact Equations

(a) If $M\,dx + N\,dy = 0$ is exact, then $\partial F/\partial x = M$. Integrate this last equation with respect to x to get

$$F(x,y) = \int M(x,y)\,dx + g(y) \ . \tag{11}$$

(b) To determine $g(y)$, take the partial derivative with respect to y of both sides of equation (11) and substitute N for $\partial F/\partial y$. We can now solve for $g'(y)$.

(c) Integrate $g'(y)$ to obtain $g(y)$ up to a numerical constant. Substituting $g(y)$ into equation (11) gives $F(x, y)$.

(d) The solution to $M\,dx + N\,dy = 0$ is given implicitly by

$$F(x,y) = C \ .$$

(Alternatively, starting with $\partial F/\partial y = N$, the implicit solution can be found by first integrating with respect to y; see Example 3.)

Example 2 Solve

$$(2xy - \sec^2 x)\,dx + (x^2 + 2y)\,dy = 0 \ . \tag{12}$$

Solution Here $M(x, y) = 2xy - \sec^2 x$ and $N(x, y) = x^2 + 2y$. Because

$$\frac{\partial M}{\partial y} = 2x = \frac{\partial N}{\partial x} \ ,$$

equation (12) is exact. To find $F(x, y)$, we begin by integrating M with respect to x:

(13) $$F(x, y) = \int (2xy - \sec^2 x)dx + g(y)$$
$$= x^2 y - \tan x + g(y) .$$

Next we take the partial derivative of (13) with respect to y and substitute $x^2 + 2y$ for N:

$$\frac{\partial F}{\partial y}(x, y) = N(x, y) ,$$
$$x^2 + g'(y) = x^2 + 2y .$$

Thus, $g'(y) = 2y$, and since the choice of the constant of integration is not important, we can take $g(y) = y^2$. Hence, from (13), we have $F(x, y) = x^2 y - \tan x + y^2$, and the solution to equation (12) is given implicitly by $x^2 y - \tan x + y^2 = C$. ◆

Remark. The procedure for solving exact equations requires several steps. As a check on our work, we observe that when we solve for $g'(y)$, we must obtain a function that is independent of x. If this is not the case, then we have erred either in our computation of $F(x, y)$ or in computing $\partial M/\partial y$ or $\partial N/\partial x$.

In the construction of $F(x, y)$, we can first integrate $N(x, y)$ with respect to y to get

(14) $$F(x, y) = \int N(x, y)dy + h(x)$$

and then proceed to find $h(x)$. We illustrate this alternative method in the next example.

Example 3 Solve

(15) $$(1 + e^x y + xe^x y)dx + (xe^x + 2)dy = 0 .$$

Solution Here $M = 1 + e^x y + xe^x y$ and $N = xe^x + 2$. Because

$$\frac{\partial M}{\partial y} = e^x + xe^x = \frac{\partial N}{\partial x} ,$$

equation (15) is exact. If we now integrate $N(x, y)$ with respect to y, we obtain

$$F(x, y) = \int (xe^x + 2)dy + h(x) = xe^x y + 2y + h(x) .$$

When we take the partial derivative with respect to x and substitute for M, we get

$$\frac{\partial F}{\partial x}(x, y) = M(x, y)$$
$$e^x y + xe^x y + h'(x) = 1 + e^x y + xe^x y .$$

Thus, $h'(x) = 1$, so we take $h(x) = x$. Hence, $F(x, y) = xe^x y + 2y + x$, and the solution to equation (15) is given implicitly by $xe^x y + 2y + x = C$. In this case we can solve explicitly for y to obtain $y = (C - x)/(2 + xe^x)$. ◆

Remark. Since we can use either procedure for finding $F(x, y)$, it may be worthwhile to consider each of the integrals $\int M(x, y)dx$ and $\int N(x, y)dy$. If one is easier to evaluate than the other, this would be sufficient reason for us to use one method over the other. [The skeptical reader should try solving equation (15) by first integrating $M(x, y)$.]

Example 4 Show that

$$(x + 3x^3 \sin y)dx + (x^4 \cos y)dy = 0 \tag{16}$$

is *not* exact but that multiplying this equation by the factor x^{-1} yields an exact equation. Use this fact to solve (16).

Solution In equation (16), $M = x + 3x^3 \sin y$ and $N = x^4 \cos y$. Because

$$\frac{\partial M}{\partial y} = 3x^3 \cos y \not\equiv 4x^3 \cos y = \frac{\partial N}{\partial x},$$

equation (16) is not exact. When we multiply (16) by the factor x^{-1}, we obtain

$$(1 + 3x^2 \sin y)dx + (x^3 \cos y)dy = 0 . \tag{17}$$

For this new equation, $M = 1 + 3x^2 \sin y$ and $N = x^3 \cos y$. If we test for exactness, we now find that

$$\frac{\partial M}{\partial y} = 3x^2 \cos y = \frac{\partial N}{\partial x},$$

and hence (17) is exact. Upon solving (17), we find that the solution is given implicitly by $x + x^3 \sin y = C$. Since equations (16) and (17) differ only by a factor of x, then any solution to one will be a solution for the other whenever $x \neq 0$. Hence the solution to equation (16) is given implicitly by $x + x^3 \sin y = C$. ◆

In Section 2.5 we discuss methods for finding factors that, like x^{-1} in Example 4, change inexact equations into exact equations.

2.4 EXERCISES

In Problems 1–8, classify the equation as separable, linear, exact, or none of these. Notice that some equations may have more than one classification.

1. $(x^{10/3} - 2y)dx + x\,dy = 0$
2. $(x^2 y + x^4 \cos x)dx - x^3\,dy = 0$
3. $\sqrt{-2y - y^2}\,dx + (3 + 2x - x^2)dy = 0$
4. $(ye^{xy} + 2x)dx + (xe^{xy} - 2y)dy = 0$
5. $xy\,dx + dy = 0$
6. $y^2\,dx + (2xy + \cos y)dy = 0$
7. $[2x + y\cos(xy)]dx + [x\cos(xy) - 2y]dy = 0$
8. $\theta\,dr + (3r - \theta - 1)d\theta = 0$

In Problems 9–20, determine whether the equation is exact. If it is, then solve it.

9. $(2x + y)dx + (x - 2y)dy = 0$
10. $(2xy + 3)dx + (x^2 - 1)dy = 0$

11. $(\cos x \cos y + 2x)dx - (\sin x \sin y + 2y)dy = 0$

12. $(e^x \sin y - 3x^2)dx + (e^x \cos y + y^{-2/3}/3)dy = 0$

13. $(t/y)dy + (1 + \ln y)dt = 0$

14. $e^t(y - t)dt + (1 + e^t)dy = 0$

15. $\cos\theta\, dr - (r \sin\theta - e^\theta)d\theta = 0$

16. $(ye^{xy} - 1/y)dx + (xe^{xy} + x/y^2)dy = 0$

17. $(1/y)dx - (3y - x/y^2)dy = 0$

18. $[2x + y^2 - \cos(x + y)]dx + [2xy - \cos(x + y) - e^y]dy = 0$

19. $\left(2x + \dfrac{y}{1 + x^2y^2}\right)dx + \left(\dfrac{x}{1 + x^2y^2} - 2y\right)dy = 0$

20. $\left[\dfrac{2}{\sqrt{1 - x^2}} + y\cos(xy)\right]dx + [x\cos(xy) - y^{-1/3}]dy = 0$

In Problems 21–26, solve the initial value problem.

21. $(1/x + 2y^2x)dx + (2yx^2 - \cos y)dy = 0$, $y(1) = \pi$

22. $(ye^{xy} - 1/y)dx + (xe^{xy} + x/y^2)dy = 0$, $y(1) = 1$

23. $(e^ty + te^ty)dt + (te^t + 2)dy = 0$, $y(0) = -1$

24. $(e^tx + 1)dt + (e^t - 1)dx = 0$, $x(1) = 1$

25. $(y^2 \sin x)dx + (1/x - y/x)dy = 0$, $y(\pi) = 1$

26. $(\tan y - 2)dx + (x\sec^2 y + 1/y)dy = 0$, $y(0) = 1$

27. For each of the following equations, find the most general function $M(x, y)$ so that the equation is exact.

(a) $M(x, y)dx + (\sec^2 y - x/y)dy = 0$

(b) $M(x, y)dx + (\sin x \cos y - xy - e^{-y})dy = 0$

28. For each of the following equations, find the most general function $N(x, y)$ so that the equation is exact.

(a) $[y\cos(xy) + e^x]dx + N(x, y)dy = 0$

(b) $(ye^{xy} - 4x^3y + 2)dx + N(x, y)dy = 0$

29. Consider the equation

$$(y^2 + 2xy)dx - x^2\,dy = 0 .$$

(a) Show that this equation is not exact.

(b) Show that multiplying both sides of the equation by y^{-2} yields a new equation that is exact.

(c) Use the solution of the resulting exact equation to solve the original equation.

(d) Were any solutions lost in the process?

30. Consider the equation

$$(5x^2y + 6x^3y^2 + 4xy^2)dx + (2x^3 + 3x^4y + 3x^2y)dy = 0 .$$

(a) Show that the equation is not exact.

(b) Multiply the equation by x^ny^m and determine values for n and m that make the resulting equation exact.

(c) Use the solution of the resulting exact equation to solve the original equation.

31. Argue that in the proof of Theorem 2 the function g can be taken as

$$g(y) = \int_{y_0}^{y} N(x, t)dt - \int_{y_0}^{y}\left[\frac{\partial}{\partial t}\int_{x_0}^{x} M(s, t)ds\right]dt ,$$

which can be expressed as

$$g(y) = \int_{y_0}^{y} N(x, t)dt - \int_{x_0}^{x} M(s, y)ds + \int_{x_0}^{x} M(s, y_0)ds .$$

This leads ultimately to the representation

$$F(x, y) = \int_{y_0}^{y} N(x, t)dt + \int_{x_0}^{x} M(s, y_0)ds . \quad (18)$$

Evaluate this formula directly with $x_0 = 0$, $y_0 = 0$ to rework

(a) Example 1.

(b) Example 2.

(c) Example 3.

32. Orthogonal Trajectories. A geometric problem occurring often in engineering is that of finding a family of curves (orthogonal trajectories) that intersects a given family of curves orthogonally at each point. For example, we may be given the lines of force of an electric field and want to find the equation